Approximation Algorithms for Geometric Problems
with Well Separated Pair Decompositions

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Context and Motivation

In what follows, we consider a finite point cloud $P \subset \mathbb{R}^3$ of size n.
We are interested in the following tasks:

- **Closest points**: find the pair $(p, q) \in P^2$ such that the distance between p and q is minimal.
- **Diameter**: find the pair $(p, q) \in P^2$ such that the distance between p and q is maximal.
- **Network visualisation**: find an embedding of a network $G = (P, E)$ in $\mathbb{R}^3$ minimising a given energy function.

All these tasks have naive solutions in time $O(n^2)$. We are looking for less time consuming approximation algorithms.
Intuition: how could we exploit the clusters in our point cloud to save time?

Well-Separated Sets

Definition

Let s be a positive real number. Two subsets $A \subset P$ and $B \subset P$ are s-**well-separated** if there exist two balls $\mathcal{B}_A$ and $\mathcal{B}_B$ of diameter r such that:

- A is a subset of $\mathcal{B}_A$: $A \subset \mathcal{B}_A$
- B is a subset of $\mathcal{B}_B$: $B \subset \mathcal{B}_B$
- The distance $d(\mathcal{B}_A, \mathcal{B}_B)$ between $\mathcal{B}_A$ and $\mathcal{B}_B$ checks: $d(\mathcal{B}_A, \mathcal{B}_B) > s \cdot r$

Well-Separated Pair Decompositions (WSPD)

Definition

Let s be a positive real number. An s-**well-separated pair decomposition** of P is a family $\{\{A_1, B_1\}, ..., \{A_m, B_m\}\}$ of pairs of subsets of P such that:

- for every $i \in \mathbb{N}$ such that $1 \leq i \leq m$, the sets A_i and B_i are well-separated.
- for every pair (p, q) of points in P, there is an index $1 \leq i \leq m$ such that $(p, q) \in A_i \times B_i$ or $(p, q) \in B_i \times A_i$

Impact of s

WSPD Implementation

Our implementation

Building an Octree from the point cloud:

- Children as Arrays vs Children as Linked Lists
- Find the size of the maximum cube and its center
- Repeatedly add the points in P, and update the parameters of the Octree

Testing Well-Separatedness

WSPD as a LinkedList vs WSPD as a HashSet

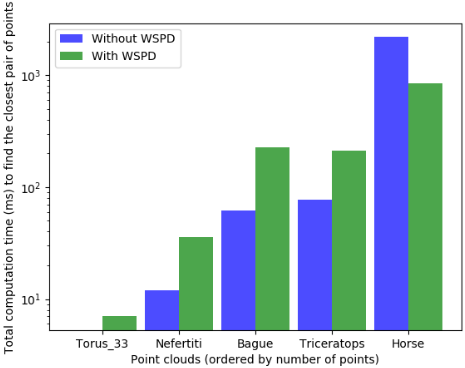
WSPDs and Closest Pair: Idea

Theorem

Let $\{\{A_1, B_1\}, ..., \{A_m, B_m\}\}$ be a well separated pair decomposition of P with parameter $s > 1$.
Then the two closest points p and q in P are such that $\{p\} = A_i$ and $\{q\} = B_i$ for some $1 \leq i \leq m$.

To compute the closest pair, it suffices to traverse the pairs of the well-separated pair decomposition that contain two singletons, and to return the such pair with smallest distance between its two points.

WSPDs and Closest Pairs: Results



Correctness: we checked that our algorithm returns the closest pair for every point cloud.
Time: though for smaller values of n the upfront cost of computing the WSPD dwarfs the time gained later, for larger point clouds such as `Horse`, this method pays off.

WSPDs and the Diameter: Idea

Theorem

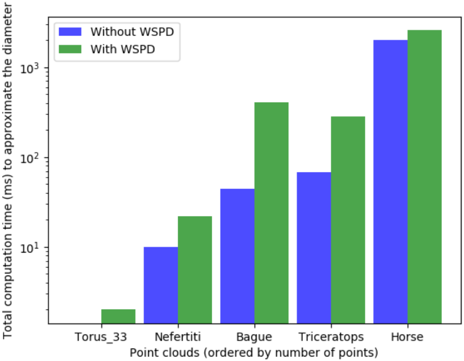
Let $P \subset \mathbb{R}^3$ be a set of points, $\epsilon > 0$ and $\{\{A_1, B_1\}, ..., \{A_m, B_m\}\}$ be a **well separated pair decomposition** *of P with parameter $s \geq \frac{2}{\epsilon}$.*
Define for every $1 \leq i \leq m$ a **representative** *α_i (respectively β_i) of A_i (respectively B_i).*
Call d_P the **diameter** *of P, the greatest distances between two points in P. Then:*

$$\boxed{\max_{1 \leq i \leq m} ||\alpha_i - \beta_i|| \geq (1 - \epsilon) \cdot d_P}$$

.

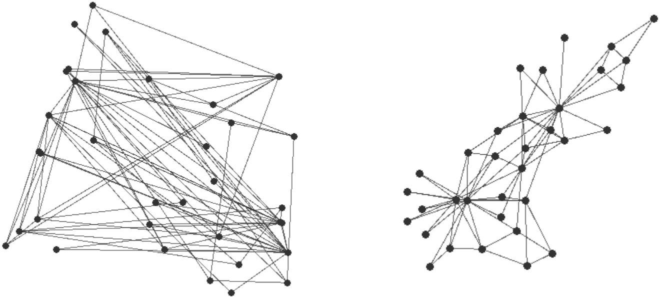
The maximum distance between representatives of the subsets of the WSPD provides us with an ϵ-approximation of the diameter of P.

WSPDs and the Diameter: Results



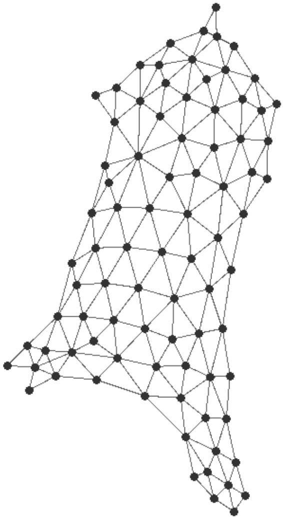
Correctness: we checked that our algorithm always returns an ϵ-approximation of the diameter of P.

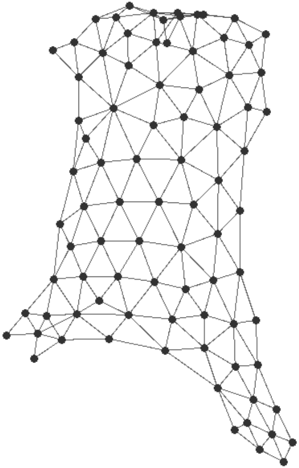
Time: this method was not very efficient to approximate the diameter: not only did it not provide us with the true diameter, it took longer than the naive $O(n^2)$ algorithm.



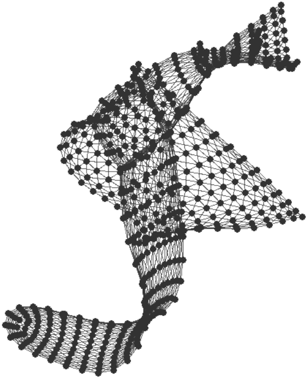
New class GraphDrawingResults to compute embeddings without displaying them

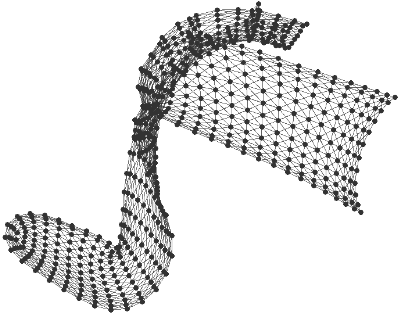
WSPDs for Network Visualisation



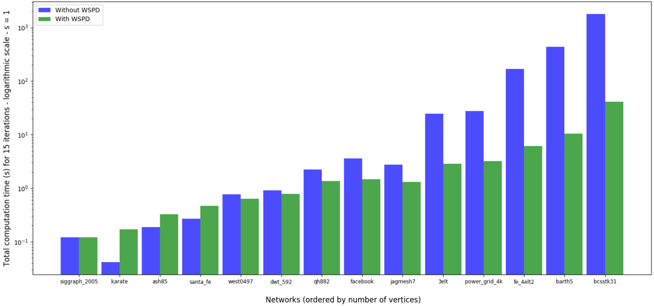


WSPDs for Network Visualisation

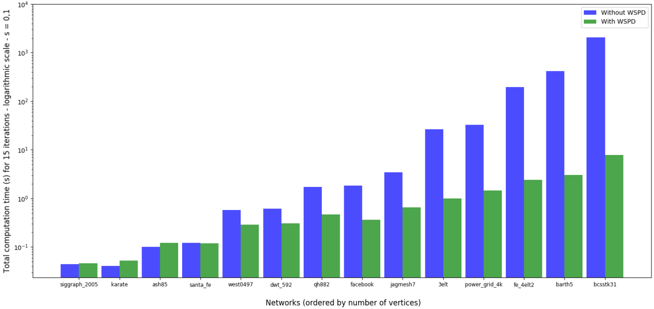




WSPDs for Network Visualisation



WSPDs for Network Visualisation



WSPDs for Network Visualisation