Irreducible Representations of the Symmetric Group

1 Partitions and λ-tableaux

Definition 1.1. (Partition) We call *partition of* n a tuple $\lambda = (\lambda_1, ...\lambda_r)$ of integers such that $\lambda_1 + ... + \lambda_r = n$ and $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_r$.

There is a natural correspondence between conjugacy classes of S_n and partitions of n: a partition $(\lambda_1, ...\lambda_r)$ is associated with the set of products of cycles of length λ_m up to λ_1.

Definition 1.2. (Young diagram) Let $\lambda = (\lambda_1, .., \lambda_r)$ be a partition of n. The *Young diagram* of λ is an array of n boxes with r rows of boxes such that the i^{th} row contains λ_i boxes.

Definition 1.3. (λ-tableau) Given a partition $\lambda = (\lambda_1, .., \lambda_r)$, a λ-*tableau* is a bijective filling of the boxes of the Young diagram of λ by the integers between 1 and n, obtained by placing each such integer in one of the boxes of the diagram.

Action of S_n on the set of λ-tableaux The symmetric group S_n acts on the set of λ-tableaux: given a tableau T and a permutation σ, if a box in the Young diagram of λ is filled by i in T then it is filled by $\sigma(i)$ in the tableau $(\sigma \cdot T)$.

2 Row stabilisers, tabloids, and the Young module

Definition 2.1. (Row stabiliser) Let T be a λ-tableau: the *row stabiliser* $R(T)$ of T is the set of permutations leaving the rows of T globally (setwise) invariant.

$R(T) \simeq S_{1,...,\lambda_1} \times S_{\lambda_1+1,...,\lambda_2} \times \ldots \times S_{\lambda_{r-1}+1,...,\lambda_r}$

Definition 2.2. (Tabloids) Two λ-tableaux T and T' are said to be row-equivalent when there is a permutation $\sigma \in R(T)$ such that $T' = \sigma \cdot T$. Row-equivalence is an equivalence relation: we call the associated equivalence classes *tabloids*. The tabloid corresponding to a tableau T as written $\{T\}$.

Since a λ-tableau is essentially the same thing as a permutation in S_n, a tabloid can be seen as an element of the quotient $S_n/R(T)$.

Lemma 2.3. (Row equivalence is S_n- invariant) *Let T and T' be row-equivalent λ-tabloids, and $\sigma \in S_n$ be a permutation. Then $\sigma \cdot T$ and $\sigma \cdot T'$ are also row equivalent.*

Proof. Let $\pi \in R(T)$ be such that $T' = \pi \cdot T$. We have $\sigma \cdot T' = \sigma\pi \cdot T = (\sigma\pi\sigma^{-1}) \cdot (\sigma \cdot T)$. Since $\pi \in R(T)$, the conjugate $\sigma\pi\sigma^{-1}$ is in $R(\sigma \cdot T)$, hence the row equivalence of $\sigma \cdot T$ and $\sigma \cdot T'$. □

Action of S_n on the set of λ-tabloids The previous lemma guarantees that the action $\sigma \cdot \{T\} = \{\sigma \cdot T\}$ of S_n on the set of tabloids is not contingent on the choice of a representative of a tabloid, and is therefore well defined.

Definition 2.4. (Young permutation module) The *Young permutation module* $M^{(\lambda)}$ is the free vector space of basis $(e_{\{T\}})_{\{T\} \in \mathcal{T}}$, where $\mathcal{T}$ is the set of all tabloids. The elements of $M^{(\lambda)}$ are the formal linear combinations $\sum_{\{T\} \in \mathcal{T}} a_{\{T\}} e_{\{T\}}$, with coefficients $a_{\{T\}} \in \mathbb{C}$.

$M^{(\lambda)}$ as a representation of S_n. The action of S_n on $\mathcal{T}$ detailed above makes the Young permutation module a semi-regular representation of S_n, via the action $\star$:

$$\forall \sigma \in S_n,\ \sigma \star \big(\sum_{\{T\} \in \mathcal{T}} a_{\{T\}} e_{\{T\}} \big) = \sum_{\{T\} \in \mathcal{T}} a_{\{T\}} e_{\sigma \cdot \{T\}} = \sum_{\{T\} \in \mathcal{T}} a_{\{T\}} e_{\{\sigma \cdot T\}}$$

This representation is far from irreducible, however (example!), hence the next section.

3 Column Stabilisers, polytabloids, and the Specht module

The column stabiliser of a tableau is defined similarly to the row stabiliser.

Definition 3.1. (Column stabiliser) Let T be a λ-tableau: the *column stabiliser* $C(T)$ of T is the set of permutations leaving the columns of T globally (setwise) invariant.

Definition 3.2. (Polytabloid) Given a λ-tableau T, the associated is an element of the Young permutation $M^{(\lambda)}$ defined as the linear combination:

$$\epsilon_T = \sum_{\pi \in C(T)} sgn(\pi) \pi \star e_{\{T\}}$$

Definition 3.3. (Specht Module) The Specht Module $S^{(\lambda)}$ of a partition λ is the subspace of $M^{(\lambda)}$ spanned by all the polytabloids: $S^{(\lambda)} = Vect\{\epsilon_T | T \in \mathbb{T}\}$.

Proposition 3.4. ($S^{(\lambda)}$ as a representation of S_n) *$S^{(\lambda)} \in M^{(\lambda)}$ is stable under the action $\star$ of S_n. Since $M^{(\lambda)}$ is a representation of S_n, $S^{(\lambda)}$ is a subrepresentation of $M^{(\lambda)}$.*

Proof. Let $\epsilon_T = \sum_{\pi \in C(T)} sgn(\pi) \pi \star e_{\{T\}} \in S^{(\lambda)}$ be a polytabloid, and $\sigma \in S_n$. Notice first that for every $\pi \in S_n$, $\pi \in C(T)$ if and only if $\sigma \pi \sigma^{-1} \in C(\sigma \cdot T)$. It follows that:

$$\begin{aligned}
\epsilon_{\sigma \cdot T} &= \sum_{\pi' \in C(\sigma \cdot T)} sgn(\sigma \pi \sigma^{-1}) \sigma \pi \sigma^{-1} \star e_{\sigma \cdot \{T\}} \\
&= \sum_{\pi \in C(T)} sgn(\pi) \sigma \pi \star e_{\{T\}} \\
&= \sigma \star \sum_{\pi \in C(T)} sgn(\pi) \pi \star e_{\{T\}} \\
&= \sigma \star \epsilon_T
\end{aligned}$$

This shows that $S^{(\lambda)}$ is a stable subspace of $M^{(\lambda)}$, and completes the proof. □

Action plan: Our goal now is to show that the irreducible representations of S_n are exactly the Specht modules $S^{(\lambda)}$, with λ spanning the partitions of n. We shall proceed in two steps:

- First, we show that the $S^{(\lambda)}$ are all **irreducible**.
- Second, we show that if λ and μ are **distinct partitions**, then the corresponding Specht module representations $S^{(\lambda)}$ and $S^{(\mu)}$ are **not equivalent**.

Recall that the **number of irreducible representations** of a group G (up to equivalence) is equal to the **number of conjugacy classes** of G. Notice that two permutations σ and τ are conjugate if and only if they can be written as the **product of cycles of equal length**. Therefore, there are as many (up to equivalence) irreducible representations of S_n as there are partitions of n. Thus, having completed the two steps above, we shall have shown that the irreducible representations of S_n are exactly the Specht modules.

4 Irreducibility of the Specht module

Lemma 4.1. *Let T and T' be two λ-tableaux. Then* $\sum_{\pi \in C(T')} sgn(\pi)\pi \star e_{\{T\}} = \epsilon_{T'}$

Proof. There exists a $\sigma \in S_n$ such that: $T = \sigma \cdot T'$. Then:

$$\begin{aligned}
\sum_{\pi \in C(T')} sgn(\pi)\pi \star e_{\{T\}} &= \sum_{\pi \in C(T')} sgn(\pi)\pi \star e_{\{\sigma \cdot T'\}} \\
&= \sum_{\pi \in C(T')} sgn(\pi)(\pi \circ \sigma) \star e_{\{T'\}} \\
&= sgn(\sigma^{-1}) \sum_{\pi \in C(T')} sgn(\pi \circ \sigma)(\pi \circ \sigma) \star e_{\{T'\}} \\
&= sgn(\sigma^{-1})\epsilon_T \\
&= \pm \epsilon_T
\end{aligned}$$

□

Lemma 4.2. *Let λ be a partition of n, W be a nontrivial subspace of $S^{(\lambda)}$, and $w \in W$. Let T be a λ-tableau. Then there exists $c \in$ such that* $\sum_{\pi \in C(T)} sgn(\pi)\pi \star w = c \cdot \epsilon_T$. *Moreover if $w \neq 0$ then $c \neq 0$.*

Proof. We first write w as a linear combination of the the basis elements of $M^{(\lambda)}$: there exist λ-tabloids $\{T_1\}, \{T_2\}, ..., \{T_m\}$ and complex numbers $a_1, a_2, ..., a_m$ such that $w = \sum_{i=1}^{m} a_i e_{\{T_i\}}$.

Then $\sum_{\pi \in C(T)} sgn(\pi)\pi \star w = \sum_{\pi \in C(T)} sgn(\pi)\pi \star \sum_{i=1}^{m} a_i e_{\{T_i\}} = \sum_{i=1}^{m} a_i \big(\sum_{\pi \in C(T)} sgn(\pi)\pi \star e_{\{T_i\}} \big)$. The previous lemma shows that for every index i in the sum, $\sum_{\pi \in C(T)} sgn(\pi)\pi \star e_{\{T_i\}} = \pm e_{\{T_i\}}$.

Therefore there is a $c \in$ such that $\sum_{\pi \in C(T)} sgn(\pi)\pi \star w = c \cdot \epsilon_T$. Finally if $w \neq 0$ then $\sum_{\pi \in C(T)} sgn(\pi)\pi \star w \neq 0$ so that $c \neq 0$. □

Theorem 4.3. *For every partition λ, the Specht module $S^{(\lambda)}$ is irreducible.*

Proof. Let W be a nontrivial S_n-invariant subspace of $S^{(\lambda)}$. Our goal here is to show that necessarily $W = S^{(\lambda)}$. Since $W \neq \{0\}$, there exists a nonzero element $w \in W$. The previous lemma shows that there is a coefficient $c \in^*$ such that $\sum_{\pi \in C(T)} sgn(\pi)\pi \star w = c \cdot \epsilon_T$.

S_n invariance shows that $\pi \star w \in W$ for every $\pi \in C(T)$ (since $C(T) \subset S_n$). Therefore $c \cdot \epsilon_T \in W$, and $\epsilon_T \in W$ because $c \neq 0$. Now let $\epsilon_{T'}$ be any basis element in $S^{(\lambda)}$: there exists a permutation $\sigma \in S_n$ such that $T' = \sigma \cdot T$. Thus $\epsilon_{T'} = \sigma \star \epsilon_T$, so that $\epsilon_{T'} \in W$ by S_n invariance. Since W is a linear subspace of $S^{(\lambda)}$ containing all the basis elements of $S^{(\lambda)}$, necessarily $W = S^{(\lambda)}$. Therefore $S^{(\lambda)}$ is an irreducible representation of S_n.

□

5 Non-equivalence of the Specht modules for distinct partitions.

Definition 5.1. We define a partial order on partitions of n by saying that a partition λ dominates another partition μ if for every index i: $\sum_{k=1}^{i} \lambda_k \geq \sum_{k=1}^{i} \mu_k$. In this case we write $\lambda \trianglerighteq \mu$.

Lemma 5.2. *Let λ and μ be partitions of n, suppose that t is a λ-tableau and s is a μ-tableau. If for every row in s the numbers in this row belong to different columns in t, then $\lambda \trianglerighteq \mu$.*

Proof. Write $\lambda = (\lambda_1, ..., \lambda_m)$ and $\mu = (\mu_1, ..., \mu_m)$ (completing the shorter of the two tuples with zeros). Suppose that for every $1 \leq i \leq m$, the numbers in row i of s are in different columns of t: applying this to the first row of s yields $\lambda_1 \geq \mu_1$.

More generally, a column of s contains at most i numbers from the first i rows of t, therefore $\lambda_1 + ... + \lambda_i \geq \mu_1 + ... + \mu_i$. □

Lemma 5.3. *Let λ and μ be partitions of n, suppose that t is a λ-tableau and s is a μ-tableau. If $\sum\limits_{\pi \in C(t)} sgn(\pi) e_{\{s\}} \neq 0$, then $\lambda \trianglerighteq \mu$.*

Proof. Suppose that we did not have $\lambda \trianglerighteq \mu$. Then there would be two integers a and b such that a and b were both in the same row in s and in the same column in t. The tabloid $e_{\{s\}}$ would therefore be invariant under composition by (ab): $(ab) \star e_{\{s\}} = e_{\{s\}}$.

Since a and b are in the same column in t, the transposition (ab) is in the column stabiliser $C(t)$, and the group generated by the identity and (ab) is as subgroup of $C(t)$. In particular, there is a subgroup H of $C(t)$ such that $C(t) = H \sqcup (ab)H$.

Thus

$$\begin{aligned}\sum_{\pi \in C(t)} sgn(\pi) e_{\pi\{s\}} &= \sum_{\sigma \in H} sgn(\sigma) e_{\sigma\{s\}} + \sum_{\sigma \in H} sgn((ab)\sigma) e_{(ab)\sigma\{s\}} \\ &= \sum_{\sigma \in H} sgn(\sigma) e_{\sigma\{s\}} - \sum_{\sigma \in H} sgn(\sigma)(ab) \star e_{\sigma\{s\}} \\ &= \sum_{\sigma \in H} sgn(\sigma) e_{\sigma\{s\}} - \sum_{\sigma \in H} sgn(\sigma) e_{\sigma\{s\}} \\ &= 0\end{aligned}$$

This contradicts our assumption that $\sum\limits_{\pi \in C(t)} sgn(\pi) e_{\{s\}} \neq 0$, therefore necessarily $\lambda \trianglerighteq \mu$.

□

Lemma 5.4. *Let λ and μ be partitions of n, and $f : S^{(\lambda)} \to S^{(\mu)}$ be a nonzero linear map commuting with the action of S_n. Then $\lambda \trianglerighteq \mu$.*

Proof. We have $M^{(\lambda)} = S^{(\lambda)} \oplus S^{(\lambda)\perp}$. Define the linear map $\tilde{f} : M^{(\lambda)} \to M^{(\mu)}$ by $\forall x \in S^{(\lambda)}, \tilde{f}(x) = f(x)$ and $\forall x \in S^{(\lambda)\perp}, \tilde{f}(x) = 0$. Then $\tilde{f}$ also commutes with the action of S_n. Since $S^{(\lambda)}$ and $S^{(\mu)}$ are irreducible and f is nonzero, Schur's lemma implies that f is a constant times the identity. Thus $ker(f) \cap S^{(\lambda)} = \{0\}$, hence $ker(\tilde{f}) \subset S^{(\lambda)\perp}$.

Let t be a λ-tableau: $\epsilon_t \in S^{(\lambda)}$ and $\epsilon_t \neq 0$, so $f(\epsilon_t) \neq 0$. Since f commutes with the action of S_n, $\sum_{\pi \in C(t)} sgn(\pi)\pi \star f(e_{\{t\}}) \neq 0$. Let $\{s_1\}, ..., \{s_m\}$ be μ-tabloids and $a_1, ..., a_m$ be complex numbers such that $f(e_{\{t\}}) = \sum_{i=1}^{m} a_i e_{\{s_i\}}$.

Then $\sum_{\pi \in C(t)} sgn(\pi)\pi \star f(e_{\{t\}}) = \sum_{i=1}^{m} a_i e_{\{s_i\}} \sum_{\pi \in C(t)} sgn(\pi)\pi \star e_{\{s_i\}}$. Since this sum is nonzero, there is at least one index $1 \leq j \leq m$ such that $\sum_{\pi \in C(t)} sgn(\pi)\pi \star e_{\{s_j\}} \neq 0$. By the previous lemma, $\lambda \trianglerighteq \mu$. ☐

Lemma 5.5. *Let λ and μ be distinct partitions of n: then the representations $S^{(\lambda)}$ and $S^{(\mu)}$ are not equivalent.*

Proof. Suppose $S^{(\lambda)}$ and $S^{(\mu)}$ were equivalent: then there would exist an isomorphism $f : S^{(\lambda)} \to S^{(\mu)}$ commuting with the action of S_n. In particular, f and f^{-1} would be nonzero and f^{-1} would also commute with the action of S_n. By the previous lemma, we would have $\lambda = \mu$, contradicting our initial assumptions. ☐

This completes the action plan above, hence the proof that **the Specht modules are exactly the irreducible representations of the symmetric group**.